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ABSOLUTELY CONTINUOUS COCYCLES OVER IRRATIONAL ROTATIONS

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ABSTRACT

For homeomorphisms

$$(z, w) \overset{T_\varphi}{\mapsto} (z \cdot e^{2\pi i\alpha}, \varphi(z)w)$$

($z, w \in \mathbf{S}^1$, α is irrational, $\varphi\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$) of the torus $\mathbf{S}^1 \times \mathbf{S}^1$ it is proved that T_φ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions whenever φ is absolutely continuous with nonzero topological degree and the derivative of φ is of bounded variation. Some other cocycles with bounded variation are studied and generalizations of the above result to certain distal homeomorphisms on finite dimensional tori are presented.

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Introduction

In the last few years, problems concerning spectral multiplicity have become of a renewed interest. There have been presented new constructions of automorphisms with given spectral multiplicity ([6], [15], [16], [17]). The history of the spectral multiplicity problem in ergodic theory till 1983 has been described in [15]. Since then, new results have appeared, especially around the Banach problem of finding an automorphism with simple Lebesgue spectrum. In 1984, Mathew and Nadkarni [11] constructed a family of automorphisms having Lebesgue component of multiplicity 2. A similar result was achieved in [12]. In [1] and [10], the authors constructed examples of automorphisms having Lebesgue component of arbitrary even multiplicities.

Let $Tz = z \cdot e^{2\pi i\alpha}$ be an irrational rotation of the circle $\mathbf{S}^1 = \{z \in \mathbf{C}\colon\ |z| = 1\}$. In this note we take up the Lebesgue spectrum problem in the class of homeomorphisms of the torus $\mathbf{S}^1 \times \mathbf{S}^1$ given by the extension

$$T_\varphi(z,w) = (z \cdot e^{2\pi i\alpha}, \varphi(z)w), \tag{1}$$

of T, where $\varphi\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ is a smooth map. Such a φ can be represented as

$$\varphi(e^{2\pi ix}) = e^{2\pi i\tilde{\varphi}(x)} \cdot e^{2\pi imx}, \tag{2}$$

where $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ is periodic of period 1 and smooth. In this representation, $m \in \mathbf{Z}$ is unique, while $\tilde{\varphi}$ is unique up to an additive integer constant. The number m is called the **degree** $d(\varphi)$ of φ.

In [5], the authors have proved that if $d(\varphi) = 0$ and $\tilde{\varphi}$ is absolutely continuous then the maximal spectral type (m.s.t.) of T_φ is singular. Here, we show that quite the opposite happens for nonzero degree and φ sufficiently smooth.

THEOREM 1: *Suppose that $\tilde{\varphi}$ is absolutely continuous and $\tilde{\varphi}'$ is of bounded variation. If $m = d(\varphi) \neq 0$, then T_φ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of T.*

In [9] (see also [3], p.344), Kushnirenko proved a similar result concerning diffeomorphisms of the form (1) under the assumption that $\tilde{\varphi}' + 1 > 0$ and $\tilde{\varphi} \in C^2(\mathbf{R})$.

According to Theorem 1 and the mentioned result of [5], on the torus $\mathbf{S}^1 \times \mathbf{S}^1$ there are no C^2-diffeomorphisms of the form (1) with Lebesgue component of finite multiplicity.

We also discuss mixing properties of (1), where φ is absolutely continuous of nonzero degree or, more generally, is piecewise absolutely continuous. In particular, we prove that if φ is absolutely continuous and has nonzero degree then in the orthocomplement of the eigenfunctions of T the automorphism (1) is mixing.

In Section 4 we show that the results obtained for absolutely continuous cocycles of nonzero degree are no longer true if we only assume bounded variation of the cocycle. This is done by a construction of a degree 1 continuous monotone cocycle which is a coboundary. The construction however requires α to have unbounded partial quotients.

In the last section, we consider more general automorphisms on finite dimensional tori defined by

$$S(e^{2\pi ix_1}, e^{2\pi ix_2}, \dots, e^{2\pi ix_q}) = (e^{2\pi i(x_1+\alpha)}, e^{2\pi i(x_2+d_{2,1}x_1+\tilde{\varphi}_1(x_1))}, \dots, e^{2\pi i(x_q+d_{q,1}x_1+\cdots+d_{q,q-1}x_{q-1}+\tilde{\varphi}_{q-1}(x_1,\dots,x_{q-1}))}), \tag{3}$$

where $d_{kn} \in \mathbf{Z}$, $d_{n,n-1} \neq 0$ for each $n = 2, \dots, q$. We generalize a result of Furstenberg [4] concerning strict ergodicity as well as a result from [3], p. 344, about the Lebesgue spectrum of such automorphisms.

1. Notation and facts from spectral theory

We assume that the reader is familiar with the basic facts on the spectral theory of unitary operators (Appendix in [13] is of sufficient scope).

Suppose that $Tz = z \cdot e^{2\pi i\alpha}$ is an irrational rotation. Denote

$$H = L^2(\mathbf{S}^1, \lambda),$$

where λ is Lebesgue measure. We will consider unitary operators $U\colon H \longrightarrow H$ given by

$$(Uf)(z) = F(z)f(Tz), \tag{4}$$

where $|F| = 1$. For each $f \in H$, we will denote by σ_f the **spectral measure** of f, i.e.

$$\hat{\sigma}_f(n) = \int_{\mathbf{S}^1} z^n d\sigma_f(z) = (U^n f, f), \quad n \in \mathbf{Z}.$$

For the operator $M\colon H \longrightarrow H$ defined by $Mf(z) = zf(z)$ the commutation relation

$$UM = e^{2\pi i\alpha} \cdot MU \tag{5}$$

holds. We have the following classical

LEMMA 1 ([13], Wiener Lemma): *Suppose that $H_0 \subseteq H$ is a closed M-invariant subspace of H. Then, there exists a Borel set B such that*

$$H_0 = \{f \in H\colon\ f \mid B^c = 0\}.$$

The next five lemmas are well-known (see [8]), but we include the proofs for the reader's convenience.

LEMMA 2: *If H_0 is a closed subspace of H which is simultaneously M- and U-invariant then $H_0 = \{0\}$ or $H_0 = H$.*

Proof: By Wiener Lemma,

$$H_0 = \{f \in H\colon\ f \mid B^c = 0\}.$$

Take $\chi_B \in H_0$. Since $U\chi_B = F \cdot \chi_{T^{-1}B} \in H_0$ and $|F| = 1$, $T^{-1}B \subseteq B$, so by the ergodicity of T either $\lambda(B) = 0$ or 1. ∎

LEMMA 3: *The m.s.t. of U is either discrete or continuous singular or Lebesgue.*

Proof: Suppose that $H = H_0 \oplus H_1 \oplus H_2$, where H_0 (H_1, H_2 resp.) consists of those $f \in H$ whose spectral measure σ_f is discrete (continuous singular, absolutely continuous resp.). Notice that H_i is a closed U-invariant subspace of H. In view of (5),

$$(U^k Mf, Mf) = e^{2\pi ik\alpha}(MU^k f, Mf) = e^{2\pi ik\alpha}(U^k f, f).$$

Hence, $\sigma_{Mf} = \sigma_f * \delta_{e^{2\pi i\alpha}}$. Consequently, each H_i is also M-invariant, so by Lemma 2 it has to be trivial.

It remains to prove that if $H_2 = H$ then the m.s.t. σ_f of U on H_2 is Lebesgue. Notice that for each $n \in \mathbf{Z}$

$$\sigma_{M^n f} = \sigma_f * \delta_{e^{2\pi in\alpha}} \ll \sigma_f \ll \lambda. \tag{6}$$

Suppose there exists a Borel set $A \subseteq \mathbf{S}^1$ such that $\sigma_f(A) = 0$ and $\lambda(A) > 0$. In view of (6),

$$\sigma_f(e^{2\pi i n\alpha}A) = 0 \quad (n \in \mathbf{Z}), \tag{7}$$

so $\sigma_f(\bigcup_{n\in\mathbf{Z}} e^{2\pi i n\alpha}A) = 0$. On the other hand, $\lambda(\bigcup_{n\in\mathbf{Z}} e^{2\pi i n\alpha}A) = 1$ by the ergodicity of the irrational rotation, a contradiction. ∎

LEMMA 4: *If the m.s.t. of U is Lebesgue then the multiplicity function of U is uniform.*

Proof: Let $H = \oplus_{n=1}^{\infty} Z(f_n)$, where

$$Z(f_n) = \text{span}\{U^i f_n : i \in \mathbf{Z}\} \quad \text{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \cdots.$$

Notice that in view of (5),

$$H = MH = \bigoplus_{n=1}^{\infty} Z(Mf_n)$$

and

$$\sigma_{Mf_1} \gg \sigma_{Mf_2} \gg \cdots$$

since $\sigma_{Mf_n} = \delta_{e^{2\pi i\alpha}} * \sigma_{f_n}$. Hence, by the uniqueness of the spectral types, we have $\sigma_{f_j} \sim \delta_{e^{2\pi i\alpha}} * \sigma_{f_j}$, $j = 1, 2, \ldots$. Therefore, all the nonzero spectral measures are equivalent to Lebesgue measure. ∎

LEMMA 5: *Suppose that $f \in H$ and $\sum_{n=-\infty}^{\infty} |(U^n f, f)|^2 < +\infty$. Then $\sigma_f \ll \lambda$.*

Proof: Let $g(z) = \sum_{k=-\infty}^{\infty} (U^k f, f) z^{-k}$ in $L^2(\mathbf{S}^1, \lambda)$. Now, the absolutely continuous measure $d\nu(z) = g(z)d\lambda(z)$ coincides with σ_f since for every $n \in \mathbf{Z}$ we have

$$\hat{\nu}(n) = \int_{\mathbf{S}^1} z^n g(z) d\lambda(z) = \sum_{k=-\infty}^{\infty} (U^k f, f) \int_{\mathbf{S}^1} z^{n-k} d\lambda(z) = (U^n f, f) = \hat{\sigma_f}(n).$$

∎

Denote

$$F^{(n)}(z) = \begin{cases} F(z)F(Tz)\cdot \ldots \cdot F(T^{n-1}z) & \text{if} \quad n > 0, \\ 1 & \text{if} \quad n = 0, \\ (F(T^n z)\cdot \ldots \cdot F(T^{-1}z))^{-1} & \text{if} \quad n < 0. \end{cases}$$

COROLLARY 1: *Suppose, for the operator U given by (4), that*

$$\sum_{n=-\infty}^{\infty} \left| \int_{\mathbf{S}^1} F^{(n)}(z) d\lambda(z) \right|^2 < +\infty. \tag{8}$$

Then U has Lebesgue spectrum of uniform multiplicity.

Proof: Put $f = 1$ and notice that (8), by Lemma 5, gives rise to the conclusion that $\sigma_f \ll \lambda$. Then, apply Lemmas 3 and 4. ∎

Now, let T_φ be given by (1). Let us decompose

$$L^2(\mathbf{S}^1 \times \mathbf{S}^1, \lambda \otimes \lambda) = \bigoplus_{n=-\infty}^{\infty} H^{(n)}, \tag{9}$$

where

$$H^{(n)} = \{g\colon\ g(z,w) = f(z)w^n,\ f \in L^2(\mathbf{S}^1, \lambda)\}.$$

Observe that $H^{(n)}$ is a closed U_{T_φ}–invariant subspace of $L^2(\mathbf{S}^1 \times \mathbf{S}^1, \lambda \otimes \lambda)$, where $U_{T_\varphi}(g) = g \circ T_\varphi$.

LEMMA 6: *The operator $U_{T_\varphi} : H^{(n)} \longrightarrow H^{(n)}$ is unitarily equivalent to $U^{(n)}$: $H \longrightarrow H$, where $(U^{(n)}f)(z) = \varphi(z)^n f(Tz)$.*

Proof: We define $V\colon H^{(n)} \longrightarrow H$ by putting $Vg = f$, where $g(z,w) = f(z)w^n$. Then V is an isometry form $H^{(n)}$ onto H and moreover

$$(U_{T_\varphi} g)(z,w) = f(Tz)(\varphi(z)w)^n = f(Tz)\varphi(z)^n w^n,$$

so

$$(VU_{T_\varphi} g)(z) = f(Tz)\varphi(z)^n = (U^{(n)}Vg)(z)$$

and the result follows. ∎

2. Proof of Theorem 1

Let $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ be periodic of period 1. Fix $\alpha \in [0,1)$ and denote

$$\tilde{\varphi}^{(n)}(x) = \begin{cases} \tilde{\varphi}(x) + \tilde{\varphi}(x+\alpha) + \cdots + \tilde{\varphi}(x+(n-1)\alpha) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(\tilde{\varphi}(x+n\alpha) + \cdots + \tilde{\varphi}(x+(-\alpha))) & \text{if } n < 0. \end{cases}$$

If $\tilde{\varphi}$ is in addition absolutely continuous then we denote by $\tilde{\varphi}'\colon \mathbf{R} \longrightarrow \mathbf{R}$ a periodic function of period 1 which is a.e. the derivative of $\tilde{\varphi}$ (then $\tilde{\varphi}$ is the indefinite integral of $\tilde{\varphi}'$).

The following lemma is similar to Lemma $(4 \cdot 3)$, due to van der Corput, from [18], p.197.

LEMMA 7: *Let $f\colon [0,1] \longrightarrow \mathbf{R}$ be absolutely continuous with $f(1) - f(0) \in \mathbf{Z}$. Let $f'\colon [0,1] \longrightarrow \mathbf{R}$ be a.e. equal to the derivative of f. Assume that the variation $\mathrm{Var}(f')$ of f' on $[0,1]$ is bounded and $f'(0) = f'(1)$. Moreover, suppose that there exists $a > 0$ such that $|f'(x)| \geq a$ for $x \in [0,1]$. Then*

$$\Big|\int_0^1 e^{2\pi i f(x)} dx\Big| \leq \frac{\mathrm{Var}(f')}{2\pi a^2}.$$

Proof: By integrating by parts we obtain

$$\begin{aligned}\Big|\int_0^1 e^{2\pi i f(x)}\, dx\Big| &= \frac{1}{2\pi}\Big|\int_0^1 \frac{d(e^{2\pi i f(x)})}{f'(x)}\Big| \\ &= \frac{1}{2\pi}\Big|\Big[\frac{e^{2\pi i f(x)}}{f'(x)}\Big]_0^1 - \int_0^1 e^{2\pi i f(x)}\, d\Big(\frac{1}{f'(x)}\Big)\Big| \\ &= \frac{1}{2\pi}\Big|\int_0^1 e^{2\pi i f(x)}\, d\Big(\frac{1}{f'(x)}\Big)\Big| \leq \frac{1}{2\pi}\mathrm{Var}\Big(\frac{1}{f'}\Big).\end{aligned}$$

Since $\mathrm{Var}(\frac{1}{f'}) \leq \frac{\mathrm{Var}(f')}{a^2}$, the result follows. ∎

LEMMA (Basic Lemma): *Suppose $\alpha \in [0,1)$ is irrational. Let $\tilde{\varphi}$ be absolutely continuous and periodic of period 1. Assume that the variation $\mathrm{Var}(\tilde{\varphi}')$ of $\tilde{\varphi}'$ on $[0,1]$ is bounded. Then, for any $m, N \in \mathbf{Z} \setminus \{0\}$*

$$\Big|\int_0^1 e^{2\pi i N(\tilde{\varphi}^{(n)}(x) + nmx)}\, dx\Big| \leq \frac{\mathrm{Var}(\tilde{\varphi}')}{|n|}$$

for $|n|$ large enough.

Proof: Fix $0 < \varepsilon < 1/2$. Since $\tilde{\varphi}'$ is Riemann integrable on $[0,1]$, from the strict ergodicity of the irrational translation

$$\Big|\frac{1}{|n|}(\tilde{\varphi}')^{(n)}(x) - \int_0^1 \tilde{\varphi}'(t)\, dt\Big| < \varepsilon$$

holds for $|n| \geq n_0$ uniformly in $x \in [0,1]$. Since $\tilde{\varphi}$ is periodic, $\int_0^1 \tilde{\varphi}'(t)\, dt = 0$. Hence $|(\tilde{\varphi}')^{(n)}(x)| < \varepsilon |n|$ for all $x \in [0,1]$ and $|n| \geq n_0$. Therefore, for $|n| \geq n_0$, we have

$$|(\tilde{\varphi}')^{(n)}(x) + nm| \geq (|m| - \varepsilon)|n|. \tag{10}$$

We also have

$$\mathrm{Var}(\tilde{\varphi}') = \mathrm{Var}(\tilde{\varphi}'(x + j\alpha)),\ j \in \mathbf{Z}, \tag{11}$$

$$(\tilde{\varphi}')^{(n)} = (\tilde{\varphi}^{(n)})'. \tag{12}$$

Put $f(x) = N(\tilde{\varphi}^{(n)}(x) + nmx)$. In view of (12) and (10), we get $|f'(x)| \geq |N|(|m|-\varepsilon)|n|$. By (11), $\mathrm{Var}(f') \leq |Nn|\mathrm{Var}(\tilde{\varphi}')$. Hence, by Lemma 7, for $|n| \geq n_0$,

$$\Big|\int_0^1 e^{2\pi i N(\tilde{\varphi}^{(n)}(x)+nmx)}\,dx\Big| \leq \frac{|Nn|\mathrm{Var}(\tilde{\varphi}')}{2\pi(N(|m|-\varepsilon)n)^2} \leq \frac{\mathrm{Var}(\tilde{\varphi}')}{|n|}$$

which completes the proof. ∎

We intend to prove that (using the notation from Lemma 6) $U_{T_\varphi}\colon H^{(N)} \longrightarrow H^{(N)}$ has uniform Lebesgue spectrum whenever $N \in \mathbf{Z} \setminus \{0\}$. In view of Lemma 6 and Corollary 1, it is enough to prove that

$$\sum_{n=-\infty}^{\infty} \Big|\int_{\mathbf{S}^1} (\varphi^{(n)}(z))^N\,d\lambda(z)\Big|^2 < +\infty.$$

Since φ is given by (2), all we need to show is that

$$\sum_{n=-\infty}^{\infty} \Big|\int_0^1 e^{2\pi i N(\tilde{\varphi}^{(n)}(x)+mnx+\frac{n(n-1)}{2}m\alpha)}\,dx\Big|^2 < +\infty,$$

which holds true by Basic Lemma. Since N runs over an infinite set, U_{T_φ} has countable Lebesgue spectrum in the orthocomplement of $H^{(0)}$ and the proof of Theorem 1 is complete. ∎

We do not know what the values of the spectral multiplicity function of $U^{(1)}$ are in case of T_φ considered in Theorem 1. The spectrum is of uniform multiplicity $|m|$ if $\tilde{\varphi}$ is constant and $d(\varphi) = m \neq 0$.

3. Remarks on absolutely continuous cocycles

Remark 1: In [5], the authors have proved that if T_φ is given by (1) and φ is absolutely continuous of nonzero topological degree then T_φ is ergodic, in fact it is weakly mixing in the orthocomplement of the eigenfunctions of T. Using our method, we can prove that T_φ is even mixing in that orthocomplement. Indeed, since T has discrete spectrum, it is enough to show that

$$\lim_{n\to\infty}\int_0^1 e^{2\pi iN(\tilde{\varphi}^{(n)}(x)+nmx)}dx=0, \tag{13}$$

for each $N,m\in\mathbf{Z}\setminus\{0\}$ and an arbitrary $\tilde{\varphi}:\mathbf{R}\longrightarrow\mathbf{R}$ which is 1-periodic and absolutely continuous. By integrating by parts, we have

$$\begin{aligned}\Big|\int_0^1 e^{2\pi iN(\tilde{\varphi}^{(n)}(x)+nmx)}dx\Big| &= \Big|\int_0^1 \frac{1}{mn}\tilde{\varphi}'^{(n)}(x)e^{2\pi iN(\tilde{\varphi}^{(n)}(x)+nmx)}dx\Big|\\ &\le \frac{1}{|m|}\int_0^1\Big|\frac{1}{n}\tilde{\varphi}'^{(n)}(x)\Big|dx.\end{aligned}$$

Since $\tilde{\varphi}'\in L^1(\mathbf{S}^1,\lambda)$, applying the ergodic theorem (L^1–convergence) to the rotation by $e^{2\pi i\alpha}$ we obtain (13). ∎

Remark 2: In the same paper [5] an isomorphism invariant $\mathcal{S}_\varphi(T)$ for the automorphisms of the form (1) has been introduced. It is given by

$$\mathcal{S}_\varphi(T)=\limsup_{\|q\alpha\|\to 0,\, q\in\mathbf{N}}\Big|\int_0^1 e^{2\pi i\varphi^{(q)}(x)}dx\Big|.$$

It has been proved in [5] that $\mathcal{S}_\varphi(T)<1$ whenever φ is uniformly Lipschitz continuous and of nonzero topological degree. From Remark 1, much more follows: the invariant is equal to zero (in fact, it is zero whenever the cocycle φ is absolutely continuous with a nonzero degree). ∎

Remark 3: A function f: $[0,1)\longrightarrow\mathbf{R}$ is said to be **piecewise absolutely continuous** if there are $0=x_0<x_1<\ldots<x_n<x_{n+1}=1$ such that f is absolutely continuous on each interval $[x_i,x_{i+1})$ (in particular $f(x_{i+1}-0)$ exists), $i=0,\ldots,n$. Notice that if f is piecewise absolutely continuous then there exist g,h: $[0,1)\longrightarrow\mathbf{R}$ such that

$$f=g-h, \tag{14}$$

where g is absolutely continuous and h is a step function, with the discontinuity points $x_1,\ldots,x_n$, so h restricted to each interval $[x_i,x_{i+1})$ is constant. If $f=g_1-h_1$ is another representation in which g_1 is absolutely continuous and h_1 a step function with the discontinuity points $y_1,\ldots,y_m$, then for some $c\in\mathbf{R}$

we have $g = g_1 + c$, $h = h_1 + c$. We say that a piecewise absolutely continuous function f is **essential** if $g(1-0)-g(0) \in \mathbf{Z} \setminus \{0\}$. Pask in [14] has proved that if f is piecewise absolutely continuous, $\int_0^1 f(t)dt = 0$ and the derivative f' is Riemann integrable with $\int_0^1 f'(t)dt \neq 0$ then for each irrational α the corresponding skew product $T_f\colon \mathbf{S}^1 \times \mathbf{R} \longrightarrow \mathbf{S}^1 \times \mathbf{R}$, $T_f(e^{2\pi i x}, s) = (e^{2\pi i(x+\alpha)}, f(x) + s)$ is ergodic (on $\mathbf{R}$ we consider the infinite Lebesgue measure). Notice that for each piecewise absolutely continuous function f its (a.e.) derivative is Lebesgue integrable and moreover

$$\int_0^1 f'(t)dt \neq 0$$

whenever f is essential. Let f be essential and $f = g - h$ be a representation (14) of f. Then the integration by parts as in Remark 1 (with m replaced by $g(1-0) - g(0)$) yields

$$\lim_{n\to\infty} \int_0^1 e^{2\pi i g^{(n)}(x)}dx = 0. \tag{15}$$

Now, h is a step function so by a result of [5] it follows that if α has unbounded partial quotients then

$$\mathcal{S}_{e^{2\pi i h}}(T) = 1. \tag{16}$$

Putting (15) and (16) together, we get that there exists a sequence (q_n), $||q_n\alpha|| \longrightarrow 0$, such that

$$\lim_{n\to\infty} \int_0^1 e^{2\pi i f^{(q_n)}(x)}dx = 0. \tag{17}$$

Indeed, this is a consequence of the following two more general observations. Let $(X, \mathcal{B}, \mu)$ be a probability space. Suppose that $(f_n), (g_n)$ are sequences of measurable functions whose values are of modulus one. Then

(i) *If* $\int_X f_n(x)d\mu(x) \longrightarrow c$, $|c| = 1$, *then* $f_n \longrightarrow c$ *in* μ
since $\int_X (1 - Re(c^{-1}f_n(x)))d\mu(x) \longrightarrow 0$ and $0 \leq 1 - Re(c^{-1}f_n(x)) \leq 2$.

(ii) *If* $f_n \longrightarrow c$ *in* μ, $|c| = 1$ *and* $\int_X g_n(x)d\mu(x) \longrightarrow 0$ *then* $\int_X f_n(x)g_n(x)d\mu(x) \longrightarrow 0$
since

$$\lim_{n\to\infty} |\int_X f_n g_n d\mu| = \lim_{n\to\infty} |\int_X f_n g_n d\mu - c\int_X g_n d\mu|$$
$$\leq \lim_{n\to\infty} \int_X |f_n - c| d\mu = 0.$$

By the result of Pask, it follows that for certain piecewise absolutely continuous maps f the skew products T_φ, where $\varphi(e^{2\pi ix}) = e^{2\pi if(x)}$, of the form (1) are ergodic. However, we can prove that in addition the transformations T_φ are weakly mixing in the orthocomplement of the eigenfunctions of T whenever f is essential. Indeed, if θ is an eigenvalue of T_φ with an eigenfunction orthogonal to $H^{(0)}$, (see (9)) then, by [2], there are a measurable function $\psi\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ and $N \in \mathbf{Z}\setminus\{0\}$ such that $\varphi^N = \theta\psi\circ T/\psi$. Hence, there exists a sequence (z_k) of complex numbers of modulus 1 such that for each $\varepsilon > 0$

$$\lim_{\|k\alpha\|\to 0} \lambda(\{z \in \mathbf{S}^1 : |(\varphi^N)^{(k)}(z) - z_k| \geq \varepsilon\}) = 0. \tag{18}$$

But the function Nf is essential so (17) is still satisfied for it. This is a contradiction to (18).

We do not know whether for every essential f and every α with bounded partial quotients the skew product $T_{e^{2\pi if}}$ is weakly mixing in the orthocomplement of the space generated by the eigenfunctions of T. ∎

4. Cocycles with bounded variation

As indicated in Remark 1, absolutely continuous cocycles with nonzero degree give rise to ergodic extensions. In 1961, Furstenberg [4] proved the above assertion under the stronger assumption of the Lipschitz property of the cocycle. He noticed that his assumption could not be essentially weakened since the result is no longer true for continuous cocycles of nonzero degree with bounded variation ([4], p.583). However, in a private conversation, Professor Furstenberg has recently communicated to us that no appropriate counterexample was ever published. This section will be devoted to constructing this kind of counterexample.

We begin with some general remarks on circle cocycles. Let $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ be an ergodic automorphism of a probability space. Let $\varphi\colon X \longrightarrow \mathbf{S}^1$ be a cocycle.

Definition 1: A set $Y \subset X$ of positive measure is called a **fixing set for** φ if for each natural number $n \geq 1$

$$\varphi^{(n)}(x) = 1 \quad \text{whenever} \quad x, T^n x \in Y, \tag{19}$$

where $\varphi^{(n)}(x) = \varphi(x)\cdot\varphi(Tx)\cdot\ldots\cdot\varphi(T^{n-1}x)$. ∎

We say that a cocycle φ is a **coboundary** if $\varphi(x) = f(Tx)/f(x)$ for a measurable function $f\colon X \longrightarrow \mathbf{S}^1$. Notice that if φ is a coboundary then the corresponding extension

$$T_\varphi\colon (X \times \mathbf{S}^1, \tilde{\mathcal{B}}, \tilde{\mu}) \longrightarrow (X \times \mathbf{S}^1, \tilde{\mathcal{B}}, \tilde{\mu}), \quad T_\varphi(x,z) = (Tx, \varphi(x)z),$$

where $\tilde{\mathcal{B}}$ is the product σ−algebra and $\tilde{\mu}$ is the corresponding product measure, is not ergodic (the function $F(x,z) = f(x)z^{-1}$ is T_φ−invariant). Actually T_φ is ergodic iff for each $k \in \mathbf{Z} \setminus \{0\}$ the cocycle φ^k is not a coboundary ([2]).

PROPOSITON 1: *If φ has a fixing set Y then φ is a coboundary.*

Proof: For an arbitrary $x \in X$ consider the set

$$g(x) = \{\varphi^{(n)}(x)\colon\ T^n x \in Y\}.$$

Since T is ergodic, $g(x)$ is nonempty: actually, under the action of T almost each point visits Y infinitely many times. Now if $T^{n_1}x, T^{n_2}x \in Y$, $n_1 < n_2$, then $T^{n_2-n_1}(T^{n_1}x) = T^{n_2}x$ so in view of (19),

$$\varphi^{(n_2)}(x) = \varphi^{(n_1)}(x)\varphi^{(n_2-n_1)}(T^{n_1}x) = \varphi^{(n_1)}(x).$$

Therefore g can be viewed as an a.e. defined function from X into $\mathbf{S}^1$. It is measurable since for any $B \subset \mathbf{S}^1$

$$g^{-1}(B) = \bigcup_{k=1}^{\infty} T^{-k}Y \cap (\varphi^{(k)})^{-1}(B).$$

Take $x \in X$ and let $T^n x \in Y$ for some $n \geq 2$. Thus $T^{n-1}(Tx) \in Y$, $n-1 \geq 1$, so $g(Tx) = \varphi(Tx) \cdot \ldots \cdot \varphi(T^{n-1}x)$ whence $\varphi(x) = g(x)/g(Tx)$ and the result follows. ∎

We will also need the following lemma.

LEMMA 9: *Let $\varepsilon, l > 0$. There exists $K(\varepsilon, l)$ such that if $K \geq K(\varepsilon, l)$ and $a < b$ with $b - a = l$ then we can find numbers*

$$a = c_0 < c_1 < \cdots < c_K = b$$

satisfying

$$|a + \frac{j \pm 1}{K} l - c_j| < \varepsilon \qquad (j = 0, 1, \ldots, K)$$

and

$$\sum_{j=0}^{K-1} c_j \in \mathbf{Z}.$$

Proof: If $l < \varepsilon$ the assertion easily follows by the continuity of summation whenever $Kl > 1$. In general, cut $[a, b]$ into consecutive small intervals of equal length, use the same K for each, and concatenate the resulting c_j's. ∎

Below, we list some properties of continued fraction expansion (see e.g. [7], Chap. X). Let α be an irrational number from $[0, 1)$ and

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

be its continued fraction expansion. The positive integers a_n are called the **partial quotients** of α. Put

$$P_0 = 0,\ P_1 = 1,\ \ P_{n+1} = a_{n+1}P_n + P_{n-1}$$

$$Q_0 = 1,\ Q_1 = a_1,\ \ Q_{n+1} = a_{n+1}Q_n + Q_{n-1}.$$

We have

$$\frac{1}{Q_n(Q_{n+1} + Q_n)} < |\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_n Q_{n+1}},$$

$$Q_{n+1}||Q_n\alpha|| + Q_n||Q_{n+1}\alpha|| = 1,$$

where $||t||$ denotes the distance of a real number t from the set of integers. By $\{t\}$ we denote the fractional part of t.

For the rest of this section we denote by T the irrational translation mod 1 by α on $[0, 1)$. Hence, from the continued fraction expansion of α we obtain, for each n, two Rokhlin towers ξ_n, $\overline{\xi}_n$ for T whose union is the whole interval $[0, 1)$. For n even

$$\xi_n = \{[0, \{Q_n\alpha\}), T[0, \{Q_n\alpha\}), \ldots, T^{(a_{n+1}Q_n + Q_{n-1})-1}[0, \{Q_n\alpha\})\},$$

$$\overline{\xi}_n = \{[\{Q_{n+1}\alpha\}, 1), T[\{Q_{n+1}\alpha\}, 1), \ldots, T^{Q_n - 1}[\{Q_{n+1}\alpha\}, 1)\}.$$

We will denote

$$I_k = [0, \{a_{2k+1}Q_{2k}\alpha\}),\ \ J_k^s = T^{sQ_{2k}}[0, \{Q_{2k}\alpha\}),$$

$s = 0, 1, \ldots, a_{2k+1} - 1$. We have

$$I_k = \bigcup_{s=0}^{a_{2k+1}-1} J_k^s$$

and $\eta_k = \{I_k, TI_k, \ldots, T^{Q_{2k}-1} I_k\}$ is a Rokhlin tower.

THEOREM 2: *If α has unbounded partial quotients then there exists a continuous increasing function $f : [0,1] \longrightarrow \mathbf{R}$, $f(1) - f(0) = 1$, and a measurable $\psi : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ such that*

$$e^{2\pi i f(x)} = \psi(e^{2\pi i(x+\alpha)})/\psi(e^{2\pi i x})$$

for a.a. $x \in [0,1)$.

Proof: Without loss of generality we may assume that the set $\{a_{2k+1} : k \geq 1\}$ is unbounded. Indeed, in passing from α to $1-\alpha$ the sequence (a_n) shifts by a single position to the left (if $a_1 > 1$) or to the right (if $a_1 = 1$) starting from $n \geq 3$. If a function f is constructed for $1-\alpha$ then $f \circ T$ works for α.

Choose $\varepsilon_j > 0$ so that

$$\sum_{j=1}^{\infty} \varepsilon_j < 1. \tag{20}$$

We will inductively define a sequence of continuous and increasing functions $f_j : [0,1] \longrightarrow [0,1]$, $f_j(0) = 0$, $f_j(1) = 1$. Moreover, for each $j \geq 1$ we will have

$$||f_{j+1} - f_j|| < \varepsilon_j.$$

The function f_j will depend on a choice of a certain subinterval Δ_j of I_{k_j} of the form $\Delta_j = J_{kj}^{u_j} \cup J_{kj}^{u_j+1} \cup \ldots \cup J_{kj}^{u_j+K_j-1}$. Denoting $\Delta_{j,s} = T^s \Delta_j$ and letting

$$B_j = \bigcup_{s=0}^{Q_{2k_j}-1} \Delta_{j,s}$$

we will have

$$\lambda(B_j) < \varepsilon_j \tag{21}$$

(here λ denotes Lebesgue measure on $[0,1)$). If, for a fixed j, we cut $[0,1)$ into the intervals $\Delta_{j,s}$ and the gaps between them, then f_j will be linear on each $\Delta_{j,s}$ and constant on each gap. If $C_j = \Delta_{j,s_1} \cup \Delta_{j,s_2} \cup \ldots \cup \Delta_{j,s_{t_j}}$ is the union of those intervals $\Delta_{j,s}$ on which f_j is nonconstant, then we will have $C_{j+1} \subset C_j$ and $f_{j+1} = f_j$ off C_j, so to build f_{j+1} we only change f_j on some of the intervals $\Delta_{j,s}$. In this induction procedure Lemma 9 will be repeatedly used to construct a fixing set for $e^{2\pi i f}$, where $f = \lim_{j\to\infty} f_j$.

STEP 1: Taking as parameters $\varepsilon_1, 1$ we apply Lemma 9 to get a number K_1. Then we find k_1 such that

$$\frac{K_1}{a_{2k_1+1}} < \varepsilon_1 \tag{22}$$

and u_1 with

$$\frac{1}{3}a_{2k_1+1} < u_1 < u_1 + K_1 - 1 < \frac{2}{3}a_{2k_1+1}.$$

Define

$$B_1 = \bigcup_{s=0}^{Q_{2k_1}-1} T^s(\Delta_1),$$

where

$$\Delta_1 = J_{k_1}^{u_1} \cup J_{k_1}^{u_1+1} \cup \cdots \cup J_{k_1}^{u_1+K_1-1}.$$

Notice that by (22), $\lambda(B_1) = Q_{2k_1}|\Delta_1| < \varepsilon_1$, so (21) holds for $j = 1$. Finally, we define $f_1\colon [0,1] \longrightarrow [0,1]$ by putting

$$f_1(x) = \begin{cases} 0 & \text{for } 0 \le x \le \beta, \\ \dfrac{x-\beta}{\gamma-\beta} & \text{for } \beta < x < \gamma, \\ 1 & \text{for } \gamma \le x \le 1, \end{cases}$$

where $\Delta_1 = [\beta, \gamma)$.

In order to illustrate the induction step we next show how to define f_2.

STEP 2: Since in Step 1 the number K_1 has been selected according to Lemma 9, we find

$$0 = f_1(\beta) = c_0 < c_1 < \cdots < c_{K_1} = f_1(\gamma) = 1$$

with

$$\sum_{k=0}^{K_1-1} c_k \in \mathbf{Z}$$

and satisfying the remaining statement of Lemma 9. Now, we apply Lemma 9 with parameters $\varepsilon_2, c_{k+1} - c_k$ to select K_2, the same for each $k = 0, 1, \ldots, K_1 - 1$. Given K_2 we choose $k_2 > k_1$ in such a way that

$$\frac{K_2}{a_{2k_2+1}} < \varepsilon_2$$

and u_2 with

$$\frac{1}{3}a_{2k_2+1} < u_2 < u_2 + K_2 - 1 < \frac{2}{3}a_{2k_2+1}.$$

We will define f_2 using the parameters c_k. More explicitly, notice that $T^{(u_1+i)Q_{2k_1}} I_{k_2}$ is a left-hand subinterval of $J_{k_1}^{u_1+i}$, $i = 0, \ldots, K_1 - 1$, and obviously

$$T^{(u_1+i)Q_{2k_1}} \Delta_2 \subset T^{(u_1+i)Q_{2k_1}} I_{k_2}.$$

Now we cut Δ_1 into the subintervals $T^{(u_1+i)Q_{2k_1}} \Delta_2$ and the gaps between them. To define f_2 set the constant values c_k on the consecutive gaps $(k = 0, 1, \ldots, K_1)$ and complete f_2 linearly on the remaining subintervals. It follows from Lemma 9 that

$$\|f_2 - f_1\| < \varepsilon_1.$$

It is also easy to see that $\lambda(B_2) < \varepsilon_2$.

INDUCTION STEP: Assume f_j has already been defined, where f_j is linearly increasing on some of the intervals $\Delta_{j,s}$ $(s = 0, 1, \ldots, Q_{2k_j} - 1)$, say,

$$\Delta_{j,s_1}, \Delta_{j,s_2}, \ldots, \Delta_{j,s_{t_j}}$$

and has constant values summing up to an integer value on the gaps between these intervals. Moreover, each interval Δ_{j,s_i} consists of K_j translates of $J_{k_j}^0$, where K_j is chosen according to Lemma 9 with parameters ε_j and $|f_j(\Delta_{j,s_i})|$ $(i = 1, 2, \ldots, t_j)$. Now, by the assertion of Lemma 9, letting $\Delta_{j,s_i} = [\beta_i, \gamma_i)$ we find

$$f_j(\beta_i) = c_{0,i} < c_{1,i} < \cdots < c_{K_j,i} = f_j(\gamma_i)$$

with

$$\sum_{k=0}^{K_j-1} c_{k,i} \in \mathbf{Z}$$

and satisfying the remaining statement of Lemma 9.

Next apply Lemma 9 with parameters ε_{j+1} and $c_{k+1,i} - c_{k,i}$ $(k = 0, 1, \ldots, K_j - 1;\ i = 1, 2, \ldots, t_j)$ to select K_{j+1}. Given K_{j+1} find $k_{j+1} > k_j$ such that

$$\frac{K_{j+1}}{a_{2k_{j+1}+1}} < \varepsilon_{j+1}$$

and u_{j+1} with

$$\frac{1}{3}a_{2k_{j+1}+1} < u_{j+1} < u_{j+1} + K_{j+1} - 1 < \frac{2}{3}a_{2k_{j+1}+1}.$$

Now define

$$\Delta_{j+1} = \bigcup_{i=0}^{K_{j+1}-1} J_{k_{j+1}}^{u_{j+1}+i}.$$

We clearly have $\lambda(B_{j+1}) < \varepsilon_{j+1}$. The components $T^s\Delta_{j+1}$ of B_{j+1} are subintervals of the corresponding translates of $I_{k_{j+1}}$. Cut each Δ_{j,s_i} into the intervals $T^s\Delta_{j+1}$, where $s = s_i + (u_j + r)Q_{2k_j}$, $r = 0, 1, \ldots, K_j - 1$, of B_{j+1} that are contained in Δ_{j,s_i} and the $K_j + 1$ gaps between them. To define f_{j+1} put the values $c_{0,i}, c_{1,i}, \ldots, c_{K_j,i}$ on the consecutive gaps (according to the natural ordering of $[0,1)$) and complete f_{j+1} linearly on the remaining intervals. Now, clearly

$$||f_{j+1} - f_j|| < \varepsilon_j$$

and f_{j+1} increases only on some of the intervals $T^s\Delta_{j+1}$, $s = 0, 1, \ldots, Q_{2k_{j+1}} - 1$, with the values of constancy (assumed between these intervals) summing up to an integer.

The description of the induction step completes the definition of $f = \lim f_j$. Now, we proceed to the second part of the proof to show $e^{2\pi i f}$ is a coboundary. Denote

$$Y = [0,1) \setminus \bigcup_{j=1}^{\infty} B_j.$$

It remains to prove that Y is a fixing set for $e^{2\pi i f}$. Let $x, T^N x \in Y$. In view of (21),(20) and Proposition 1, all we have to show is that

$$f(x) + f(Tx) + \ldots + f(T^{N-1}x) \in \mathbf{Z}. \tag{23}$$

First note that if $Tx, T^2x, \ldots, T^{N-1}x \in Y$ then $f(x), f(Tx), \ldots, f(T^{N-1}x) \in \{0,1\}$ in which case we are done. Therefore we may assume $T^n x$ is not in Y for some $0 < n < N$ and let n be minimal with this property. Now $f(x), f(Tx), \ldots, f(T^{n-1}x) \in \{0,1\}$ and there exists $j \geq 1$ such that $T^n x \in B_j$. Since $T^{n-1}x$ is not in B_j, which is a Rokhlin tower with base Δ_j, we have $T^n x \in \Delta_j$. As B_j can also be viewed as a Rokhlin tower with base $J_{k_j}^{u_j}$ and height $Q_{2k_j}K_j$, we must have

$$T^n x \in J_{k_j}^{u_j}.$$

We are going to prove

$$f(T^n x) + f(T^{n+1}x) + \cdots + f(T^{n+Q_{2k_j}K_j-1}x) \in \mathbf{Z}. \tag{24}$$

By the inductive construction above, the function f_j is constant on some of the components $T^s\Delta_j$ of B_j and increases linearly on the other components $\Delta_{j,s_1}, \Delta_{j,s_2}, \ldots, \Delta_{j,s_{t_j}}$. Moreover, $f = f_j$ on the components of constancy, and the values assumed on these components sum up to an integer M_j (note that these are all the constant values assumed by f_j). Now, (24) is equal to the number M_jK_j plus those summands $f(T^sx)$ for which $T^sx \in \Delta_{j,s_1} \cup \Delta_{j,s_2} \cup \cdots \cup \Delta_{j,t_j}$, so to prove (24) it remains to show that these summands add up to an integer.

First, we observe that T^nx is not in B_{j+1}. Indeed, T^nx cannot be in Δ_{j+1} since $\Delta_{j+1} \subset I_{k_{j+1}} \subset J^0_{k_j}$ and $J^0_{k_j} \cap \Delta_j = \emptyset$. On the other hand, if $T^nx \in B_{j+1} \setminus \Delta_{j+1}$ then $T^{n-1}x \in B_{j+1}$ so $T^{n-1}x$ would not belong to Y, a contradiction.

Now, split $J^{u_j}_{k_j}$ into three consecutive subintervals A^1_j, A^2_j, A^3_j with $A^2_j = T^{Q_{2k_j}u_j}\Delta_{j+1} \subset B_{j+1}$. Note that $T^nx \in A^1_j \cup A^3_j$ and consequently $f(T^{n+l}x) = f_{j+1}(T^{n+l}x)$ for $l = rQ_{2k_j} + s_i$ $(r = 0, 1, \ldots, K_j - 1, i = 1, 2, \ldots, t_j)$. We consider two cases

CASE 1: $T^nx \in A^1_j$. Now, for each $i = 1, 2, \ldots, t_j$

$$T^{n+s_i}x \in T^{s_i}A^1_j \subset \Delta_{j,s_i}$$

so the sum of those $f(T^sx)$ in (24) for which $T^sx \in \Delta_{j,s_i}$ is equal to

$$c_{0,i} + c_{1,i} + \cdots + c_{K_j-1,i} \in \mathbf{Z}.$$

CASE 2: $T^nx \in A^3_j$. We have

$$T^{n+s_i}x \in T^{s_i}A^3_j \subset \Delta_{j,s_i}$$

so the sum of those $f(T^sx)$ in (24) for which $T^sx \in \Delta_{j,s_i}$ is equal to

$$c_{1,i} + c_{2,i} + \cdots + c_{K_j,i}. \tag{25}$$

There exists a permutation σ of $\{1, 2, \ldots, t_j\}$ such that the disjoint intervals

$$\Delta_{j,s_{\sigma(1)}}, \Delta_{j,s_{\sigma(2)}}, \ldots, \Delta_{j,s_{\sigma(t_j)}}$$

follow the natural ordering of $[0,1)$. Note that $c_{0,\sigma(1)} = 0$, $c_{K_j,\sigma(t_j)} = 1$, and $c_{K_j,\sigma(i)} = c_{0,\sigma(i+1)}$ for $i = 1, 2, \ldots, t_j - 1$. By adding up the partial sums (25) corresponding to $i = 1, 2, \ldots, t_j$ we obtain

$$\begin{aligned}\sum_{i=1}^{t_j}\sum_{k=1}^{K_j} c_{k,i} &= \sum_{i=1}^{t_j}\sum_{k=1}^{K_j} c_{k,\sigma(i)} \\ &= c_{0,\sigma(1)} + \sum_{i=1}^{t_j-1}\sum_{k=1}^{K_j} c_{k,\sigma(i)} + \sum_{k=1}^{K_j-1} c_{k,\sigma(t_j)} + c_{K_j,\sigma(t_j)} \\ &= \sum_{i=1}^{t_j}\sum_{k=0}^{K_j-1} c_{k,\sigma(i)} + 1 \in \mathbf{Z}.\end{aligned}$$

Thus far we have proved (24), which also yields

$$f(x) + f(Tx) + \cdots + f(T^{n+Q_{2k_j}K_j-1}x) \in \mathbf{Z}.$$

Observe that if we denote $x_1 = T^{n+Q_{2k_j}K_j}x$ and $N_1 = N - (n + Q_{2k_j}K_j)$ then $1 \leq N_1 < N$ and $T^{N_1}x_1 \in Y$ so the remaining part of (23) is equal to

$$f(x_1) + f(Tx_1) + \cdots + f(T^{N_1-1}x_1).$$

Now if $x_1 \in Y$, we may repeat the same argument for x_1, N_1 in place of x, N to prove that the last sum is an integer; we will be done after a finite number of steps.

To complete the proof we show $x_1 \in Y$. Since I_{k_j} is disjoint with $B_1 \cup B_2 \cup \cdots \cup B_{j-1}$ and $x_1 \in I_{k_j}$, the point x_1 cannot belong to $B_1 \cup B_2 \cup \cdots \cup B_{j-1}$. Suppose $x_1 \in B_{j+r}$ for some $r \geq 1$. Since $\Delta_{j+r} \cap B_j = \emptyset$ and clearly x_1 is not in Δ_{j+r}, we have $T^{-k}x_1 \in B_{j+r}$ for all $k = 0, 1, \ldots, p$, where p is a number greater than $Q_{2k_j}K_j$. In particular,

$$T^{-(Q_{2k_j}K_j+1)}x_1 = T^{n-1}x \in B_{j+r}$$

whence $T^{n-1}x \notin Y$, a contradiction. Since obviously x_1 does not belong to B_j we must have $x_1 \in Y$, which ends the proof of the theorem. ∎

5. Some generalization

Let $T: (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a standard Borel space. Fix a cocycle $\varphi: X \longrightarrow \mathbf{S}^1$. Assume also that a measurable function $g: X \times \mathbf{R} \longrightarrow \mathbf{R}$ is periodic of period 1 and absolutely continuous as a function on $\mathbf{R}$ for all $x \in X$. Let $g'(x,\cdot)$ be a periodic function of period 1 which is a.e. equal to the derivative of $g(x,\cdot)$. Moreover, assume that

$$g' \in L^1(X \times [0,1]).$$

For a fixed $r \in \mathbf{Z} \setminus \{0\}$ put

$$\psi(x, e^{2\pi i s}) = e^{2\pi i (g(x,s)+rs)},$$

where $s \in [0,1)$. Denote by $\tilde{\tilde{\mu}}$ the product measure on $X \times \mathbf{S}^1 \times \mathbf{S}^1$.
For an automorphism $\tau\colon (Y,\mathcal{C},\nu) \longrightarrow (Y,\mathcal{C},\nu)$ and a function $u\colon Y \longrightarrow \mathbf{R}$ we set

$$S_n(u(y),\tau) = u(y) + \cdots + u(\tau^{n-1}y),$$

while for $\xi\colon Y \longrightarrow \mathbf{C}$ we put

$$\Pi_n(\xi(y),\tau) = \xi(y) \cdot \ldots \cdot \xi(\tau^{n-1}y).$$

Under the above assumptions we have the following result.

THEOREM 3: *The automorphism $(T_\varphi)_\psi\colon X \times \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow X \times \mathbf{S}^1 \times \mathbf{S}^1$ defined by*

$$(T_\varphi)_\psi(x, e^{2\pi i s}, e^{2\pi i t}) = (Tx, \varphi(x)e^{2\pi i s}, \psi(x, e^{2\pi i s})e^{2\pi i t})$$

is mixing on the orthocomplement $\mathcal{H}$ of the functions depending only on the first two coordinates in $L^2(\tilde{\tilde{\mu}})$.

Proof: There exists a measurable function $f\colon X \longrightarrow \mathbf{R}$ such that $\varphi(x) = e^{2\pi i f(x)}$. Let $T_f\colon X \times \mathbf{R} \longrightarrow X \times \mathbf{R}$ be defined by $T_f(x,s) = (Tx, f(x)+s)$. We then have

$$\begin{aligned}\Pi_n(\psi(x,e^{2\pi i s}),T_\varphi) &= \Pi_n(e^{2\pi i(g(x,s)+rs)},T_\varphi) = e^{2\pi i S_n(g(x,s)+rs,T_f)} \\ &= e^{2\pi i S_n(g(x,s),T_f)} e^{2\pi i r n s} e^{2\pi i r \sum_{j=1}^{n-1} S_j(f(x),T)}.\end{aligned}$$

Let

$$G(x,w,z) = F(x)w^M z^N,$$

where F is bounded, $|F| \le C$, $M \in \mathbf{Z}$, $N \in \mathbf{Z} \setminus \{0\}$. Denote

$$v_n = \int_X \int_{\mathbf{S}^1} \int_{\mathbf{S}^1} G(((T_\varphi)_\psi)^n(x,w,z))\overline{G(x,w,z)}\, d\mu(x)\, d\lambda(w)\, d\lambda(z).$$

Since the functions G form a linearly dense set of functions in $\mathcal{H}$, all we need to show is that $\lim_{n\to\infty} v_n = 0$. We have

$$v_n = \int_X \int_{\mathbf{S}^1} \int_{\mathbf{S}^1} G(T^n x, \Pi_n(\varphi(x), T)w, \Pi_n(\psi(x,w), T_\varphi)z)\overline{G(x,w,z)}$$
$$d\mu(x)\, d\lambda(w)\, d\lambda(z)$$
$$= \int_X \int_{\mathbf{S}^1} F(T^n x)\overline{F(x)}(\Pi_n(\varphi(x),T))^M (\Pi_n(\psi(x,w),T_\varphi))^N \, d\mu(x)\, d\lambda(w)$$
$$= \int_X \int_0^1 F(T^n x)\overline{F(x)}(\Pi_n(\varphi(x),T))^M (\Pi_n(\psi(x,e^{2\pi i s}),T_\varphi))^N \, d\mu(x)\, ds$$
$$= \int_X F(T^n x)\overline{F(x)}(\Pi_n(\varphi(x),T))^M e^{2\pi i r N \sum_{j=1}^{n-1} S_j(f(x),T)}$$
$$\times \Big(\int_0^1 e^{2\pi i N S_n(g(x,s),T_f)} e^{2\pi i N r n s}\, ds\Big)\, d\mu(x).$$

By integrating by parts, we obtain that

$$\begin{aligned} |v_n| &\le C^2 \int_X |[\frac{e^{2\pi i N r n s}}{2\pi i N r n} e^{2\pi i N S_n(g(x,s),T_f)}]_0^1 \\ &\qquad - \int_0^1 \frac{e^{2\pi i N r n s}}{2\pi i N r n} e^{2\pi i N S_n(g(x,s),T_f)} 2\pi i N S_n(g'(x,s),T_f)\, ds|\, d\mu(x) \\ &= \frac{C^2}{\pi |Nrn|} + \frac{C^2}{|r|}\int_X \int_0^1 |\frac{1}{n} S_n(g'(x,s),T_f)|\, ds\, d\mu(x) \\ &= \frac{C^2}{\pi |Nrn|} + \frac{C^2}{|r|}\int_X \int_0^1 |\frac{1}{n} S_n(h(x,e^{2\pi i s}),T_\varphi)|\, ds\, d\mu(x), \end{aligned}$$

where $h(x,e^{2\pi i s}) = g'(x,s)$, $(x \in X,\ s \in \mathbf{R})$. Since $g'(x,\cdot) \in L^1(X \times [0,1])$,

$$\int_X \int_0^1 h(x, e^{2\pi i s})\, ds\, d\mu(x) = \int_X (g(x,1) - g(x,0))\, d\mu(x) = 0,$$

so $\int_X \int_0^1 |\frac{1}{n} S_n(h(x,e^{2\pi i s}),T_\varphi)|\, ds\, d\mu(x) \longrightarrow 0$ by the ergodicity of T_φ, whence $v_n \longrightarrow 0$. ∎

Remark 4: It follows from Theorem 3 that the automorphism $(T_\varphi)_\psi$ is ergodic. Hence, we generalize a result from [4] concerning the strict ergodicity of certain homeomorphisms of the form $(T_\varphi)_\psi$ because the uniform Lipschitz condition assumed there guarantees that the derivatives are bounded a.e., thus in L^1. By another statement from [4] (Thm.2.1) the transformations (3) are strictly ergodic whenever each $\tilde{\varphi}_j$ satisfies uniform Lipschitz condition in x_j. Theorem 3 says more, namely

COROLLARY 2: *Let S be given by (3). If for each $j = 1, \dots, q-1$ the function $\tilde{\varphi}_j(x_1, \dots, x_{j-1}, \cdot)$ is absolutely continuous with the (a.e.) derivative in L^1 then S is mixing on the orthocomplement in $L^2(\mathbf{S}^1 \times \cdots \times \mathbf{S}^1, \lambda \otimes \cdots \otimes \lambda)$ of the functions depending on the first coordinate.*

Proof: We apply inductively Theorem 3 to the transformations $(\dots(T_{\varphi_1})_{\varphi_2}\dots)_{\varphi_j}$, where

$$Tz = e^{2\pi i\alpha}z, \quad \varphi_j(e^{2\pi i x_1}, \dots, e^{2\pi i x_j}) = e^{2\pi i(x_j + d_{j,1}x_1 + \cdots + d_{j,j-1}x_{j-1} + \tilde{\varphi}_{j-1}(x_1, \dots, x_{j-1}))},$$

$j = 1, \dots, q-1$. ∎

Using the method introduced in the proof of Basic Lemma we easily extend the result obtained in Theorem 1 to the following.

THEOREM 4: *Under the assumptions of Theorem 3 if in addition the functions $g(x, \cdot)$ have derivatives of bounded variation (uniformly in x) then $(T_\varphi)_\psi$ has absolutely continuous spectrum in the orthocomplement of the functions depending on the first two coordinates.*

Proof: By the proof of Theorem 3 and Lemma 5 all we need is show that

$$\int_X \Big| \int_0^1 e^{2\pi i N S_n(g(x,s),T_f)} e^{2\pi i r n N s}\, ds \Big|\, d\mu(x) = O\Big(\frac{1}{n}\Big).$$

This follows as in Remark 1. ∎

Consequently, we obtain the following strenghtening of a result from [3], p. 344.

COROLLARY 3: *If S is given by (3) and the functions $\tilde{\varphi}_j(x_1, \dots, x_{j-1}, \cdot)$ have derivatives of bounded variation (uniformly in $x_1, \dots, x_{j-1}$) then S has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions for the irrational rotation.*

Proof: From Theorem 4 it follows that S has absolutely continuous spectrum in the orthocomplement of the eigenfunctions of T. But as T_{φ_1} has uniform infinite Lebesgue spectrum in that orthocomplement, the same holds for S. ∎

Added in June 1992: Independently of this paper, G.H. Choe has proved Theorem 1 assuming that the cocycle $\tilde{\varphi}$ belongs to $C^2(\mathbf{R})$.

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